BROWNIAN PARTICLES INTERACTING THROUGH THEIR RANKS

INVARIANT MEASURE OF GAPS

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Domaines - *Fields*

Probabilités
Probability

Analyse complexe
Complex analysis

Théorie de Galois
Galois theory

Mots-clés - *Keywords*

Invariants de Tutte
Tutte's invariant

Mesure stationnaire
Stationary distribution

Transformée de Laplace
Laplace transform

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RESYST
Reflected Stochastic Systems

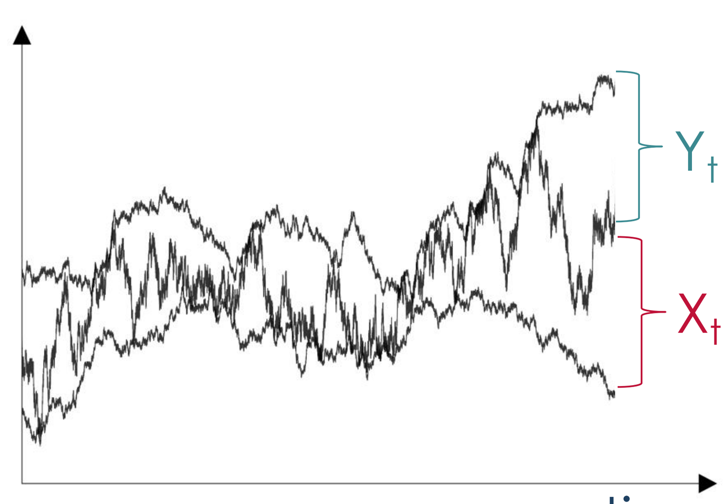
Lien vers la prépublication
Link to preprint

arXiv

CONTEXT

THE MODEL

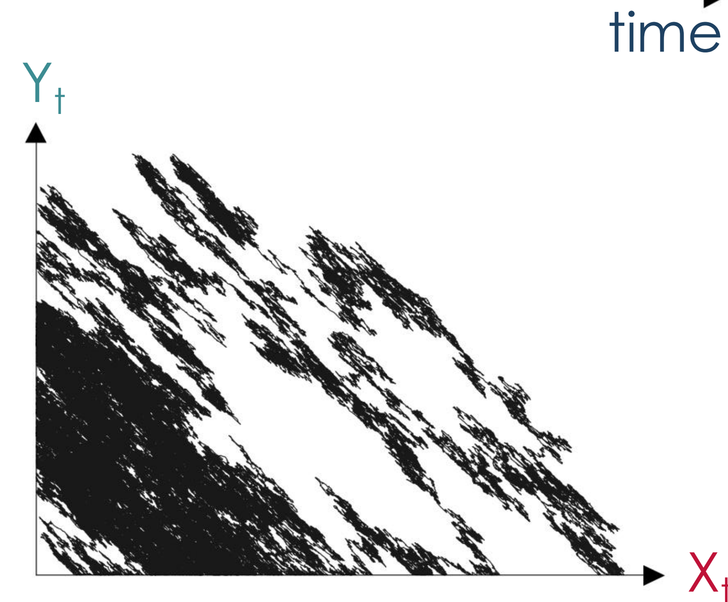
Applications in finance, physics,...



- Three particles with **rank-dependent** drift and variance.
- At any time t, denote
  X_t = *middle – bottom*
  Y_t = *top – middle*
- The "gap process" (X_t, Y_t) is a **RBM**, *i.e.* a

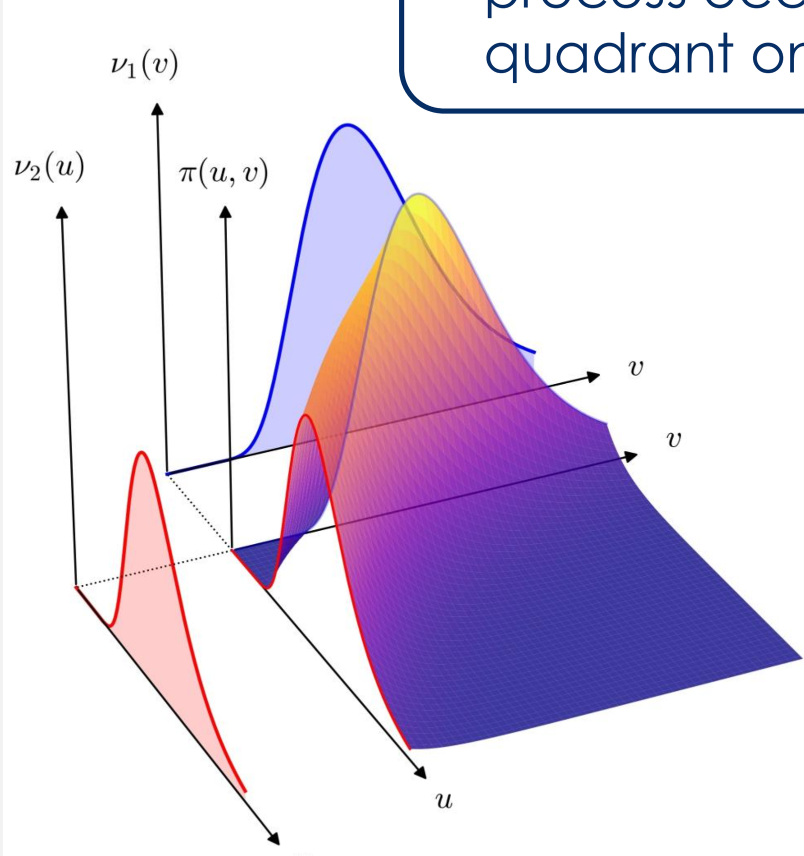
*R*eflected *B*rownian *M*otion

- Under some geometric conditions, it is **recurrent**



It visits any region of the quadrant infinitely often

? How does the gap process occupy the quadrant on average?



According to its **stationary distribution** π !

Densities	›››	*Laplace transforms*
$\pi(u,v)$	›››	$\varphi(x,y)$
$\nu_1(v) = \pi(0,v)$	›››	$\varphi_1(y)$
$\nu_2(u) = \pi(u,0)$	›››	$\varphi_2(x)$

The Laplace transforms φ, φ_1 and φ_2 satisfy the following functional equation

$$K\varphi + k_1\varphi_1 + k_2\varphi_2 = 0$$

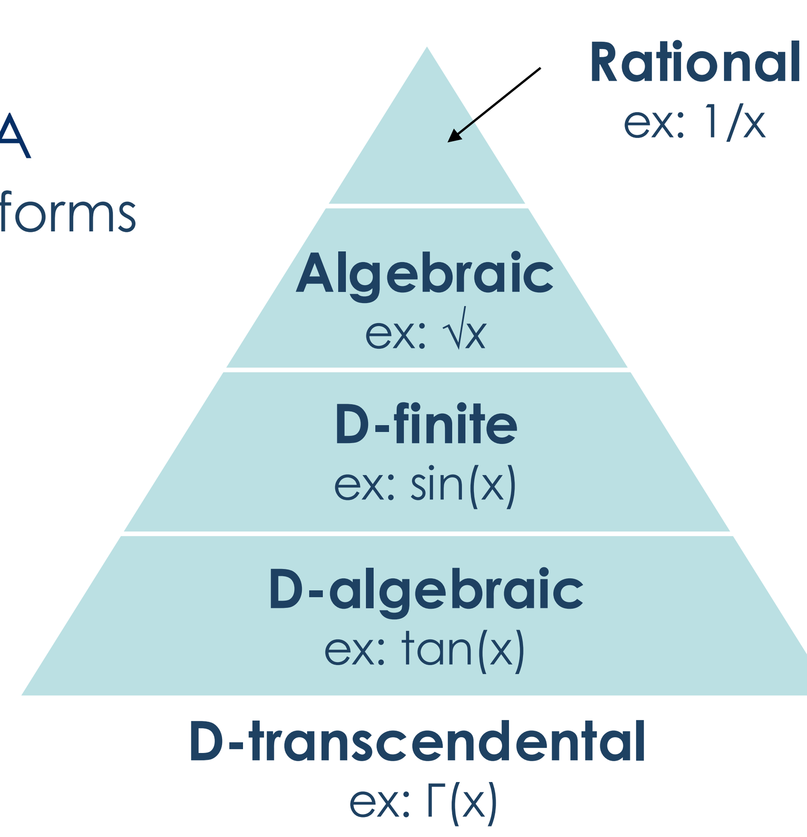
degree 2 polynomial

PROBLEM STATEMENT

1. EXPLICIT FORMULA for the Laplace transforms

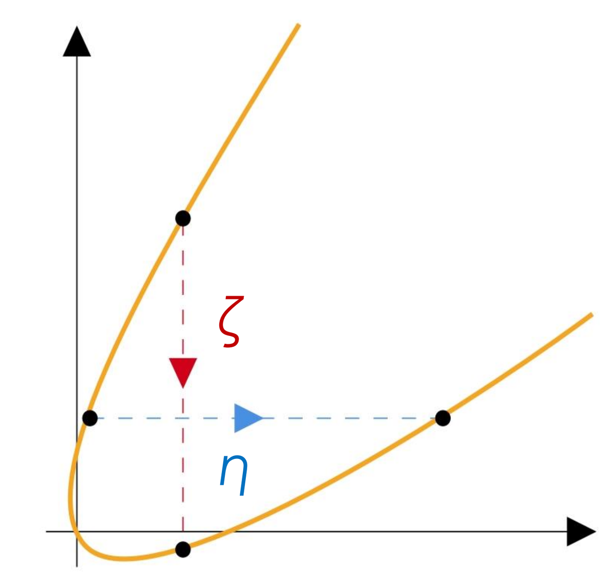
2. CLASSIFICATION

Position the Laplace transforms within a hierarchy of functions based on the type of ODE they solve.



3. INVERSION of the Laplace transform φ_1 to recover the lateral measure ν_1

TUTTE'S INVARIANT METHOD



- *Parametrize $\{K = 0\}$ by a single complex variable s*
- *Define the automorphisms η and ζ*
- *Use parametrization, automorphisms and functional equation to show that*

$$\varphi_1(s+1) = G(s)\varphi_1(s)$$

- *Find a function D such that $G(s) = \frac{D(s+1)}{D(s)}$ so that $f := \varphi_1/D$ satisfies*

$$f(\eta s) = f(\zeta s) = f(s)$$ ← "Invariant"

- *Find a **canonical invariant** w and show that every invariant is a rational function of w.*

NEW RESULTS

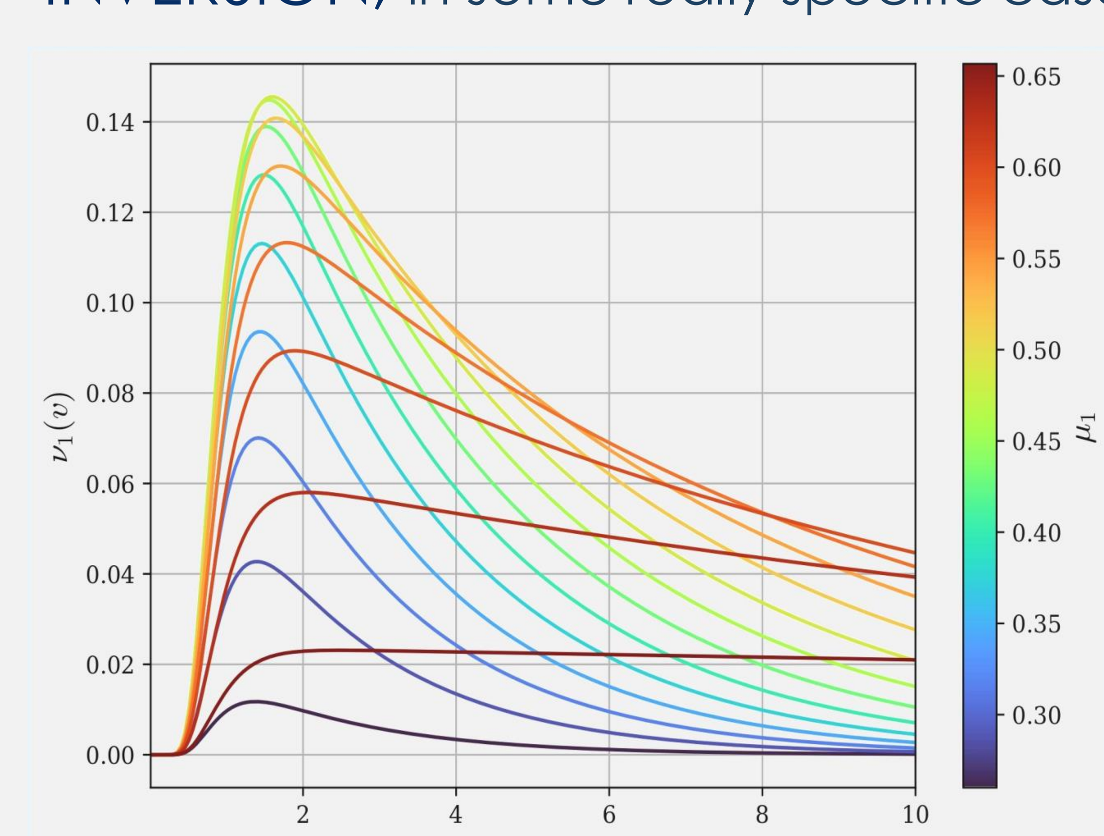
When the variances are allowed to be 0,

1. EXPLICIT FORMULA, for the Laplace transforms, for **any choice of parameters**

2. **Complete** CLASSIFICATION, depending on three *new* quantities γ, γ_1 and γ_2

nature of ϕ_1 ϕ_2 and ϕ	rational	algebraic	D-finite	D-algebraic	D-transcendental
condition	$\gamma \in -\mathbb{N}$			$\gamma \in \mathbb{Z}$ or $\{\gamma_1, \gamma_2\} \subset \mathbb{Z}$	$\gamma \notin \mathbb{Z}$ and $\{\gamma_1, \gamma_2\} \not\subset \mathbb{Z}$

3. INVERSION, in some really specific cases



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