Cramér-Rao Bound for Stationary and Cyclostationary Spherical Invariant Random Processes

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- In parametric estimation, any measurements $x = (x_1, .., x_n)$ are not *independent identically distributed* (i.i.d.), especially in signal and array processing. Typical non-iid assumptions: *stationary or cyclostationary processes* are valid for radar, sonar, and communication signals.
- The *Fisher Information Matrix* (FIM) and *Cramér-Rao Bound* (CRB) quantify the best achievable estimation accuracy.
- For zero-mean purely non-deterministic stationary Gaussian processes, the *Whittle formula* [1] provides an asymptotic closed-form expression of the FIM thanks to an ML approach.

[1] P. Whittle, "The analysis of multiple stationary time series," *Journal of the Royal Statistical Society*, vol. 15, no. 1, 1953.

- In practice, many signals are non-Gaussian due to heavy-tailed clutter or impulsive noise. Extending CRB results to *Spherically Invariant Random Processes* (SIRP) [2] allows robust performance analysis in real-world conditions.

[2] K. Yao, "A representation theorem and its applications to spherically invariant random processes," *IEEE Trans. Info. Theory*, vol. 19, no. 5, Sept. 1973.

Elliptically symmetric (ES) distributions of r.v

- **Discrete time Gaussian process:**
  A process is $\{\mathbf{x}_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}^m$ i.f.f. the stacked vector $\mathbf{x} = (\mathbf{x}_{k_1}^T, .., \mathbf{x}_{k_n}^T)^T \sim \mathcal{N}_{mn}(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x)$, $\forall k_1 <, .., < k_n \in \mathbb{Z}$, $\forall n \in \mathbb{N}^*$, i.e., its p.d.f. is

$$p(\mathbf{x}) = (2\pi)^{-mn/2}|\boldsymbol{\Sigma}_x|^{-1/2}\exp[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_x)^T\boldsymbol{\Sigma}_x^{-1}(\mathbf{x}-\boldsymbol{\mu}_x)]$$

- **Generalization to elliptically symmetric distributions:**
  Can we define *random processes* whose finite-dimensional distributions are more general than Gaussian, that is, elliptically symmetric (ES) $\mathbf{x} \sim \mathcal{E}(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x, g_{mn}^{\boldsymbol{\alpha}})$?

$$p(\mathbf{x}) = |\boldsymbol{\Sigma}_x|^{-1/2} g_{mn}^{\boldsymbol{\alpha}}[(\mathbf{x}-\boldsymbol{\mu}_x)^T\boldsymbol{\Sigma}_x^{-1}(\mathbf{x}-\boldsymbol{\mu}_x)]$$

  - $\mathrm{E}(\mathbf{x}) = \boldsymbol{\mu}_x$, $\mathrm{Cov}(\mathbf{x}) = \boldsymbol{\Sigma}_x$, $\boldsymbol{\alpha}$ nuisance parameter
  - $g_{mn}^{\boldsymbol{\alpha}}(.) \in \mathcal{G}$: family of density generators parameterized by $\boldsymbol{\alpha}$.

- **Stochastic representation**:

$$\mathbf{x} =_d \boldsymbol{\mu}_x + \sqrt{\mathcal{Q}_{mn}}\boldsymbol{\Sigma}_x{}^{1/2}\mathbf{u}_{mn}$$

  $\mathcal{Q}_{mn}$ is a positive r.v., $\mathbf{u}_{mn}$ is uniformly distributed on the unit mn-sphere, and $\mathcal{Q}_{mn}$ and $\mathbf{u}_{mn}$ are independent.

$$\mathcal{Q}_{mn} \sim p(q) = \frac{\pi^{mn/2}}{\Gamma(mn/2)} q^{mn/2-1} g^{\boldsymbol{\alpha}}_{mn}(q).$$

- **Marginal and independence properties:**
  For $\mathbf{x} = (\mathbf{x}_1^T, \mathbf{x}_2^T)^T \in \mathbb{R}^{mn}$, $\mathbf{x}_i \in \mathbb{R}^{m_i}$, $m_1 + m_2 = mn$
  - if $\mathbf{x}_{m_1} \sim \mathcal{E}(\boldsymbol{\mu}_{x_1}, \boldsymbol{\Sigma}_{x_1}, g^{\boldsymbol{\alpha}}_{m_1})$ and $\mathbf{x}_{m_2} \sim \mathcal{E}(\boldsymbol{\mu}_{x_2}, \boldsymbol{\Sigma}_{x_2}, g^{\boldsymbol{\alpha}}_{m_2})$, and $\mathbf{x}_{m_1}$ and $\mathbf{x}_{m_2}$ independent $\Rightarrow$ $\mathbf{x}$ *not ES distributed.*
  - If $\mathbf{x} \sim \mathcal{E}(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x, g^{\boldsymbol{\alpha}}_{mn})$: *Marginals*: $\mathbf{x}_{m_1} \sim \mathcal{E}(\boldsymbol{\mu}_{x_1}, \boldsymbol{\Sigma}_{x_1}, g^{\boldsymbol{\alpha}}_{m_1/mn})$ with $g^{\boldsymbol{\alpha}}_{m_1/mn}(.) \notin \mathcal{G}$ or $\in \mathcal{G}$

$$g^{\boldsymbol{\alpha}}_{m_1/mn}(u) = \frac{\pi^{m_2/2}}{\Gamma(m_2/2)} \int_u^{+\infty} (t-u)^{m_2/2-1} g^{\boldsymbol{\alpha}}_{mn}(t)dt.$$

Kolmogorov permutation and marginal consistency conditions

- $g^{\boldsymbol{\alpha}}_{m_1/mn}(.) \in \mathcal{G}$ means $g^{\boldsymbol{\alpha}}_{m_1/mn}(.) = g^{\boldsymbol{\alpha}}_{m_1}(.)$

Equivalent characterizations

- *Integral (mixture) representation:* $\exists$ positive r.v. τ with c.d.f. $F^{\boldsymbol{\alpha}}_{\tau}(.)$ s.t. [2]
  $g^{\boldsymbol{\alpha}}_{mn}(t) = (2\pi)^{-mn/2} \int_0^{\infty} \tau^{-mn/2} \exp(-t/2\tau) dF^{\boldsymbol{\alpha}}_{\tau}(\tau)$

  [2] K. Yao, "A representation theorem and its applications to spherically invariant random processes," *IEEE Trans. Info. Theory*, vol. 19, no. 5, Sept. 1973.

- *Stochastic representation:* [3] $\mathbf{x} =_d \boldsymbol{\mu}_x + \sqrt{\tau}\boldsymbol{\Sigma}_x{}^{1/2}\mathbf{n}_{mn}$, $\mathbf{n}_{mn} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_{mn})$.

  [3] G.L. Wise and N.C. Gallagher, "On spherically invariant random processes," *IEEE Trans. Info. Theory*, vol. 24, no. 1, Jan. 1978.

- $\exists$ positive r.v. τ s.t. $\mathcal{Q}_{mn} =_d \tau\chi^2_{mn}$

These processes $(\mathbf{x}_k)_{k\in\mathbb{Z}} \in \mathbb{R}^m$ are called *spherically invariant random processes* (SIRP), *compound-Gaussian (CG) processes* or *scale mixtures of normal processes.*

Stationary and cyclostationary SIRP (1)

- **Definition:** A SIRP is stationary [resp., purely cyclostationary with cycle period p] if the joint distribution of the r.v. $(\mathbf{x}_{1+k}, ..., \mathbf{x}_{n+k})$ [resp., $(\mathbf{x}_{1+kp}, ..., \mathbf{x}_{n+kp})$] is identical to that of $(\mathbf{x}_1, ..., \mathbf{x}_n)$ for all $n, k \in \mathbb{Z}$.
- **Wide-sense equivalence:** A SIRP is (cyclo)stationary i.f.f. it is wide-sense (cyclo)stationary.
- **Stationary** $\iff$ $\mathrm{E}(\mathbf{x}_\ell)$ and $\mathrm{E}(\mathbf{x}_\ell \mathbf{x}_{\ell+k}^T)$ independent of $\ell, \forall k \in \mathbb{Z}$

  Covariance structure: $\mathbf{\Sigma}_{\mathbf{y}_n} \overset{\text{def}}{=} \mathrm{Cov}(\mathbf{y}_n)$, where $\mathbf{y}_n \overset{\text{def}}{=} (\mathbf{x}_1^T, ..., \mathbf{x}_n^T)^T \in \mathbb{R}^{mn} \;\; \forall n \in \mathbb{N}^*$ is *symmetric positive definite block-Toeplitz* with block size m. Its (k, ℓ)-th block is given by $\mathbf{R}_x(\ell - k)$ where the sequence $\mathbf{R}_x(k) \overset{\text{def}}{=} \mathrm{E}(\mathbf{x}_\ell \mathbf{x}_{\ell+k}^T) \in \mathbb{R}^{m \times m}$ is assumed to be *absolutely summable*. The spectrum of $(\mathbf{x}_k)_{k \in \mathbb{Z}}$ is then given by

$$\mathbf{S}_x(f) \overset{\text{def}}{=} \sum_k \mathbf{R}_x(k) e^{-i2\pi k f}.$$

- **Cyclostationarity with cycle period** p i.f.f. $\mathrm{E}(\mathbf{x}_{k+ip})$ and $\mathrm{E}(\mathbf{x}_{k+ip}\mathbf{x}_{\ell+ip}^T)$ is independent of $i \in \mathbb{Z}$, $\forall k, l \in \mathbb{Z}$.
- $\mathbf{\Sigma}_{\mathbf{y}_n} \stackrel{\text{def}}{=} \mathrm{Cov}(\mathbf{y}_n)$, where
  $\mathbf{y}_n \stackrel{\text{def}}{=} (\underbrace{\mathbf{x}_1^T, .., \mathbf{x}_p^T}_{\mathbf{x}_1'^T}, .., \underbrace{\mathbf{x}_{(k-1)p+1}^T, .., \mathbf{x}_{kp}^T}_{\mathbf{x}_k'^T}, .., \underbrace{\mathbf{x}_{n-p+1}^T, .., \mathbf{x}_n^T}_{\mathbf{x}_q'^T})^T \in \mathbb{R}^{mn}$
  for $n = pq$ is *symmetric positive definite block-Toeplitz* with block size mp. Its (k, ℓ)-th block is given by $\mathbf{R}_{x'}(\ell - k)$ where the sequence $\mathbf{R}_{x'}(k) \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{x}_\ell' \mathbf{x}_{\ell+k}'^T) \in \mathbb{R}^{mp \times mp}$ is assumed to be *absolutely summable*. The spectrum of $(\mathbf{x}_k')_{k \in \mathbb{Z}}$ is also given by
  $$\mathbf{S}_x'(f) \stackrel{\text{def}}{=} \sum_k \mathbf{R}_{x'}(k) e^{-i2\pi k f} \in \mathbb{C}^{mp \times mp}$$
- **Cyclic spectra** at frequencies $0, 1/p, ...(p-1)/p$:
  $\mathrm{E}(\mathbf{x}_\ell \mathbf{x}_{\ell+\tau}^T) = \sum_{k=0}^{p-1} \mathbf{R}_x^{(\frac{k}{p})}(\tau) e^{i2\pi k\ell/p}$ periodic in ℓ with period p
  $$\mathbf{S}_x^{(\frac{k}{p})}(f) \stackrel{\text{def}}{=} \sum_{\tau \in \mathbb{Z}} \mathbf{R}_x^{(\frac{k}{p})}(\tau) e^{-i2\pi\tau f} \in \mathbb{C}^{m \times m}, k = 0, 1, .., p-1.$$

For the r.v. $\mathbf{y}_n \stackrel{\text{def}}{=} (\mathbf{x}_1^T, .., \mathbf{x}_n^T)^T \sim \mathcal{E}(\boldsymbol{\mu}_{y_n}, \boldsymbol{\Sigma}_{y_n}, g_{mn}^{\boldsymbol{\alpha}})$ where $\boldsymbol{\mu}_{y_n}, \boldsymbol{\Sigma}_{y_n}$ are parameterized by $\boldsymbol{\theta} \in \mathbb{R}^q$. The three FIM:

$\mathbf{I}_{y_n}(\boldsymbol{\theta}) = \mathbf{I}_{\boldsymbol{\theta}}$ for $g_{mn}(.)$ known [4]
$\bar{\mathbf{I}}_{y_n}(\boldsymbol{\theta}|g)$ for $g_{mn}(.)$ unknown (semiparametric FIM) [5]
$\bar{\mathbf{I}}_{y_n}(\boldsymbol{\theta}|\boldsymbol{\alpha}) = \mathbf{I}_{\boldsymbol{\theta}} - \mathbf{I}_{\boldsymbol{\theta},\boldsymbol{\alpha}} \mathbf{I}_{\boldsymbol{\alpha}}^{-1} \mathbf{I}_{\boldsymbol{\theta},\boldsymbol{\alpha}}^T$ for $g_{mn}^{\boldsymbol{\alpha}}(.)$ known up to $\boldsymbol{\alpha}$ [6]

share the same structure with the same value for $a_{0,mn}$ and $a_{1,mn}$ but with different values for $a_{2,mn}$.

$$
\begin{aligned}
[\text{FIM}_{y_n}(\boldsymbol{\theta})]_{k,\ell} \quad = \quad & a_{0,mn} \boldsymbol{\mu}_{y_n,k}^{\prime T} \boldsymbol{\Sigma}_{y_n}^{-1} \boldsymbol{\mu}_{y_n,\ell}^{\prime} \\
& + a_{1,mn} \text{Tr}(\boldsymbol{\Sigma}_{y_n}^{-1} \boldsymbol{\Sigma}_{y_n,k}^{\prime} \boldsymbol{\Sigma}_{y_n}^{-1} \boldsymbol{\Sigma}_{y_n,\ell}^{\prime}) + a_{2,mn} \text{Tr}(\boldsymbol{\Sigma}_{y_n}^{-1} \boldsymbol{\Sigma}_{y_n,k}^{\prime}) \text{Tr}(\boldsymbol{\Sigma}_{y_n}^{-1} \boldsymbol{\Sigma}_{y_n,\ell}^{\prime})
\end{aligned}
$$

[4] O. Besson and Y. I. Abramovich, “On the Fisher information matrix for multivariate elliptically contoured distributions,” *IEEE Signal Process. Lett.*, vol. 20, no. 11, Nov. 2013.
[5] S. Fortunati, et al, “Semiparametric CRB and Slepian-Bangs formulas for complex elliptically symmetric distributions,” *IEEE Trans. Signal Process.*, vol. 67, no. 20, Oct. 2019.
[6] H. Abeida and J.-P. Delmas, "Slepian-Bangs formulas for parameterized density generator of elliptically symmetric distributions," *Signal Processing*, vol. 205, Jan. 2023.

where $a_{0,mn} = \xi_{1,mn}$ and $a_{1,mn} = \frac{1}{2}\xi_{2,mn}$ with

$$\xi_{1,mn} \stackrel{\text{def}}{=} \frac{\mathrm{E}[Q_{mn}\varphi^2(Q_{mn})]}{mn} \quad \text{and} \quad \xi_{2,mn} \stackrel{\text{def}}{=} \frac{\mathrm{E}[Q_{mn}^2\varphi^2(Q_{mn})]}{mn(mn+2)},$$

with $\varphi(t) \stackrel{\text{def}}{=} -\frac{2}{g_{mn}(t)}\frac{dg_{mn}(t)}{dt}$ and

$$\begin{aligned}
a_{2,mn}^{\text{Clas}} &= \frac{1}{4}(\xi_{2,mn} - 1) \text{ for } g \text{ known} \\
a_{2,mn}^{\text{SePa}} &= -\frac{1}{2mn}\xi_{2,mn} \text{ for } g \text{ unknown} \\
a_{2,mn}^{\text{Pa}} &= \frac{1}{4}(\xi_{2,mn} - 1) - \boldsymbol{\xi}_{3,mn}^T \Xi_{4,mn}^{-1} \boldsymbol{\xi}_{3,mn} \text{ for } g \text{ known up to } \boldsymbol{\alpha}
\end{aligned}$$

$\boldsymbol{\xi}_{3,mn} \stackrel{\text{def}}{=} \frac{\mathrm{E}[Q_{mn}\varphi(Q_{mn})\boldsymbol{\varphi}^{\boldsymbol{\alpha}}(Q_{mn})]}{mn}$, $\Xi_{4,mn} \stackrel{\text{def}}{=} \mathrm{E}[\boldsymbol{\varphi}^{\alpha}(Q_{mn})\boldsymbol{\varphi}^{\boldsymbol{\alpha}T}(Q_{mn})]$
where $\boldsymbol{\varphi}^{\boldsymbol{\alpha}}(t) \stackrel{\text{def}}{=} -\frac{1}{g_{mn}^{\boldsymbol{\alpha}}(t)}\frac{\partial g_{mn}^{\boldsymbol{\alpha}}(t)}{\partial \boldsymbol{\alpha}}$.

Slepian-Bangs formula (3)

Properties of $a_{0,mn}$, $a_{1,mn}$ and $a_{2,mn}$

- For fixed n
  - For Gaussian v.a.
    $(a_{0,mn}, a_{1,mn}, a_{2,mn}^{\text{Clas}}, a_{2,mn}^{\text{SePa}}) = (1, \frac{1}{2}, 0, -\frac{1}{2mn})$
  - For arbitrary ES v.a.
    $a_{0,mn} \geq 1$,
    $a_{2,mn}^{\text{SePa}} \leq a_{2,mn}^{\text{Pa}} \leq a_{2,mn}^{\text{Clas}}$
  - For arbitrary CG v.a.
    $0 \leq a_{1,mn} \leq 1/2$ and $a_{2,mn}^{\text{Clas}} \leq 0$
- Proving the asymptotic behavior of $(a_{0,mn}, a_{1,mn}, a_{2,mn})$ for $n \to \infty$ for CG distributed r.v. seems possible on a case-by-case basis because:

$$\xi_{i,mn} \propto \int_0^\infty \int_0^\infty u^i v^i \left(\frac{\int_0^\infty \tau^{-mn/2-1} \exp(-uv/2\tau) dF_\tau(\tau)}{\int_0^\infty \tau^{-mn/2} \exp(-uv/2\tau) dF_\tau(\tau)} \right)^2 dF_\tau(v) p_{\chi^2_{mn}}(u) du, \quad i = 1, 2.$$

Asymptotic FIM: Whittle formula (1)

- Examples are given in [7] for the ϵ *contaminated Gaussian distribution* and the *Student t distribution* of degree of freedom $\nu > 2$.
- We deduce
  $\lim_{n\to\infty} a_{0,mn} = c_0 \geq 1$,
  $\lim_{n\to\infty} a_{1,mn} = 1/2$
  $\lim_{n\to\infty} n(a_{2,mn}^{\mathrm{SePa}}, a_{2,mn}^{\mathrm{Pa}}, a_{2,mn}^{\mathrm{Clas}}) = (-1/2, c_2^{\mathrm{Pa}}, c_2^{\mathrm{Clas}})$ with $-1/2 \leq c_2^{\mathrm{Pa}} \leq c_2^{\mathrm{Clas}} \leq 0$.
- If the general existence of these limits cannot be proved, we use the Result: $\lim_{n\to\infty} \frac{1}{n}\mathrm{FIM}_{y_n}(\boldsymbol{\theta})$ exists for stationary processes with finite Markov order (i.e., when there exists [8] $q \in \mathbb{N}^*$ such that: $p(\mathbf{x}_n/\mathbf{x}_{n-1}, \mathbf{x}_{n-2}, ...) = p(\mathbf{x}_n/\mathbf{x}_{n-1}, ..., \mathbf{x}_{n-q})$.

[7] J.-P. Delmas and H. Abeida, "Generalization of Whittle's formula to compound-Gaussian processes," *IEEE Signal Process. Letters* vol. 31, 2024.
[8] M. Radaelli, et al, "Fisher information of correlated stochastic processes," *New Journal of Physics*, June 2023.

- To consider the first and second order terms of the Slepian-Bang formula, we have used asymptotic properties of sequences of block-Toeplitz matrices based on the concept of *asymptotically equivalent sequences* introduced in [9] and extended in [10]:
  For d_n^1, d_n^2 strictly increasing, $d_n^1 \times d_n^2$ matrices $\mathbf{A}_n$ and $\mathbf{B}_n$
  $\mathbf{A}_n \sim \mathbf{B}_n \stackrel{\text{def}}{=}$
  $\|\mathbf{A}_n\|_2, \|\mathbf{B}_n\|_2 \leq M < \infty$ and $\lim_{n\to\infty} \frac{\|\mathbf{A}_n - \mathbf{B}_n\|_{\text{Fro}}}{\sqrt{n}} = 0$
  applied to the *block Toeplitz* $\mathbf{\Sigma}_{y_n} = \mathbf{T}_{mn}(\mathbf{S}_x(f))$ matrix and an associated *block circulant* matrix $\mathbf{C}_{mn}(\mathbf{S}_x(f))$.
  and the additional hypothesis $\mathbf{S}_x(f)$ is non-singular for all f.

[9] R. M. Gray, *Toeplitz and Circulant Matrices: A Review*, The essence of knowledge, Found. Trends Commun. Inf. Theory, vol. 2, no. 3, 2006.
[10] J. Gutierrez-Gutierrez and P. M. Crespo, *Block Toeplitz Matrices:Asymptotic Results and Applications*, The essence of knowledge, Found. Trends Commun. Inf. Theory, vol. 8, no. 3, 2012.

Asymptotic FIM: Whittle formula (3)

- Proved limits:

$$\lim_{n\to\infty}\frac{1}{n}\underbrace{[\boldsymbol{\mu}_{x,k}^{\prime T},..,\boldsymbol{\mu}_{x,k}^{\prime T}]}_{\boldsymbol{\mu}_{y_n,k}^{\prime T}}\underbrace{\boldsymbol{\Sigma}_{y_n}^{-1}}_{mn\times mn}[\boldsymbol{\mu}_{x,\ell}^{\prime T},...,\boldsymbol{\mu}_{x,\ell}^{\prime T}]^T=\boldsymbol{\mu}_{x,k}^{\prime T}\underbrace{\mathbf{S}_x^{-1}(0)}_{m\times m}\boldsymbol{\mu}_{x,\ell}^{\prime}.$$

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{Tr}(\boldsymbol{\Sigma}_{y_n}^{-1}\boldsymbol{\Sigma}_{y_n,k}^{\prime}\boldsymbol{\Sigma}_{y_n}^{-1}\boldsymbol{\Sigma}_{y_n,\ell}^{\prime})=\int_0^1\mathrm{Tr}(\mathbf{S}_x^{-1}(f)\mathbf{S}_{x,k}^{\prime}(f)\mathbf{S}_x^{-1}(f)\mathbf{S}_{x,\ell}^{\prime}(f))df$$

$$\lim_{n\to\infty}\frac{1}{n^2}\mathrm{Tr}(\boldsymbol{\Sigma}_{y_n}^{-1}\boldsymbol{\Sigma}_{y_n,k}^{\prime})\mathrm{Tr}(\boldsymbol{\Sigma}_{y_n}^{-1}\boldsymbol{\Sigma}_{y_n,\ell}^{\prime})$$

$$=\left(\int_0^1\mathrm{Tr}(\mathbf{S}_x^{-1}(f)\mathbf{S}_{x,k}^{\prime}(f))df\right)\left(\int_0^1\mathrm{Tr}(\mathbf{S}_x^{-1}(f)\mathbf{S}_{x,\ell}^{\prime}(f))df\right)$$

Asymptotic FIM: Whittle formula (4)

Main result:
Under the previous assumptions, the *FIM rate limit* for multidimensional *stationary* SIRP, has the following expression:

$$\lim_{n\to\infty} \frac{1}{n} \left(\mathrm{FIM}_{y_n}(\boldsymbol{\theta})\right)_{k,\ell} = c_0 \boldsymbol{\mu}'^{T}_{x,k} \mathbf{S}_x^{-1}(0) \boldsymbol{\mu}'_{x,\ell}$$
$$+ \quad \frac{1}{2} \int_0^1 \mathrm{Tr}(\mathbf{S}_x^{-1}(f) \mathbf{S}'_{x,k}(f) \mathbf{S}_x^{-1}(f) \mathbf{S}'_{x,\ell}(f)) df$$
$$+ c_2 \left(\int_0^1 \mathrm{Tr}(\mathbf{S}_x^{-1}(f) \mathbf{S}'_{x,k}(f)) df \right) \left(\int_0^1 \mathrm{Tr}(\mathbf{S}_x^{-1}(f) \mathbf{S}'_{x,\ell}(f)) df \right)$$

where $c_2 = (-1/2, c_2^{\mathrm{Pa}}, c_2^{\mathrm{Clas}})$. For multidimensional *purely cyclostationary* SIRP, $(c_0,\ \frac{1}{2},\ c_2)$, $\mathbf{S}_x(f)$ and $\mathbf{S}'_{x,k}(f)$ are replaced by $(\frac{c_0}{p^2},\ \frac{1}{2p},\ \frac{c_2}{p^2})$, $\mathbf{S}_{x'}(f)$ and $\mathbf{S}'_{x',k}(f)$, respectively.

with $\boldsymbol{\mu}'_{x,k} \stackrel{\mathrm{def}}{=} \frac{\partial \boldsymbol{\mu}_x}{\partial \theta_k}$ and $\mathbf{S}'_{x,k}(f) \stackrel{\mathrm{def}}{=} \frac{\mathbf{S}_x(f)}{\partial \theta_k}$.

Asymptotic FIM: Whittle formula (5)

For cyclostationary SIRP, the spectrum $\mathbf{S}_{x'}(f) \in \mathbb{C}^{mp \times mp}$ has limited interpretability, unlike the p cyclic spectra $\mathbf{S}_x^{(\frac{k}{p})}(f) \in \mathbb{C}^{m \times m}, k = 0, 1, .., p-1$ for which we have proved the following result:

Result: Under the previous assumptions, the *FIM rate limit* for zero-mean multidimensional *cyclostationary* SIRP, has the following expression:

$$\lim_{n\to\infty} \frac{1}{n}(\mathrm{FIM}_{y_n}(\boldsymbol{\theta}))_{k,\ell} = \frac{1}{2p}\int_0^1 \mathrm{Tr}(\mathbf{S}_x^{\mathrm{cyc}^{-1}}(f)\mathbf{S}_{x,k}^{\mathrm{cyc}'}(f)\mathbf{S}_x^{\mathrm{cyc}^{-1}}(f)\mathbf{S}_{x,\ell}^{\mathrm{cyc}'}(f))df$$
$$+\frac{c_2}{p^2}\left(\int_0^1 \mathrm{Tr}(\mathbf{S}_x^{\mathrm{cyc}^{-1}}(f)\mathbf{S}_{x,k}^{\mathrm{cyc}'}(f)df\right)\left(\int_0^1 \mathrm{Tr}(\mathbf{S}_x^{\mathrm{cyc}^{-1}}(f)\mathbf{S}_{x,\ell}^{\mathrm{cyc}'}(f))df\right)$$

where $\mathbf{S}_x^{\mathrm{cyc}}(f) \in \mathbb{C}^{mp \times mp}$ is the so-called *poly-spectral matrix* [11] whose (k', ℓ')th blocks are $\frac{1}{p}\mathbf{S}_x^{(\frac{\ell'-k'}{p})}\left(\frac{f-k'+1}{p}\right)$, $k', \ell' = 1, ..., p$.

[11] A.R. Nematollahi and T. Subba Rao, "On the spectral density estimation of periodically correlated (cyclostationary) time series," *The Indian Journal of Statistics*, vol. 67, no. 3, 2005.

Application to DOA estimation (1)

All the previous results extend to (circular and noncircular) complex-valued SIRP using $\mathbb{R}^{2m}$ to $\mathbb{C}^m$ complex representations.

- Stationary DOA model:
  $\mathbf{R}_x(k) = \mathbf{A}\mathbf{R}_s(k)\mathbf{A}^H + \sigma_n^2 \delta_{k,0}\mathbf{I}_m \overset{\text{FT}}{\to} \mathbf{S}_x(f) = \mathbf{A}\mathbf{S}_s(f)\mathbf{A}^H + \sigma_n^2\mathbf{I}_m$
- Single source case:
  $\mathbf{A} = \mathbf{a}_1(\omega_1)$, $S_s(f) = \sigma_{s_1}^2 S_{a_1}(f)$ with $\int_0^1 S_{a_1}(f)df = 1$,
  $r_1 \overset{\text{def}}{=} \sigma_{s_1}^2/\sigma_n^2$. $\boldsymbol{\theta} = (\omega_1, a_1, \sigma_{s_1}^2, \sigma_n^2)^T$.
  α_1 is purely geometric factor.
- ω_1 and $(a_1, \sigma_{s_1}^2, \sigma_n^2)$ are decoupled in the asymptotic FIM rate.

$$\lim_{n\to\infty} n\text{CRB}_c(\omega_1) = \frac{1}{\alpha_1 r_1 \int_0^1 \frac{m r_1 S_{a_1}^2(f)}{1+m r_1 S_{a_1}(f)} df}$$

$$\underbrace{\frac{1}{\alpha_1 r_1}}_{\text{Deterministic CRB}} < \lim_{n\to\infty} n\text{CRB}_c(\omega_1) \leq \underbrace{\frac{1}{\alpha_1 r_1}\left(1 + \frac{1}{m r_1}\right)}_{\text{i.i.d. Gaussian Stochastic CRB}}$$

- Two sources case:
  $\mathbf{A} = [\mathbf{a}_1(\omega_1), \mathbf{a}_2(\omega_2)]$, $\mathbf{S}_s(f) = \mathrm{Diag}(S_{s_1}(f), S_{s_2}(f))$ with $S_{s_1}(f) = \frac{\sigma_{s_1}^2}{a} \mathbb{1}_{[0,a]}(f)$, $S_{s_2}(f) = \frac{\sigma_{s_2}^2}{a} \mathbb{1}_{[1-a,1]}(f)$ with $a \in (0, 1]$ controls the overlapping. $\boldsymbol{\theta} = (\omega_1, \omega_2, a, \sigma_{s_1}^2, \sigma_{s_2}^2, \sigma_n^2)^T$

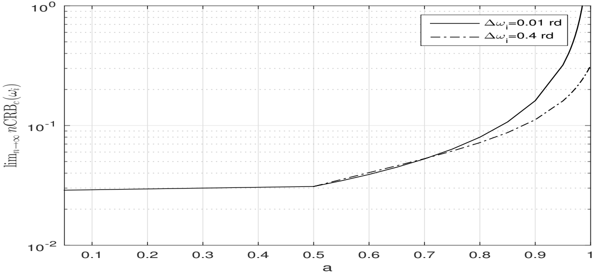

Figure: $\lim_{n\to\infty} n\mathrm{CRB}_c(\omega_i)$ *as a function of* a *for different values of* $\Delta\omega$ *for SNR = 0dB for two equipowerered circular sources impinging on a uniform linear array of 6 sensors.*

Conclusion

- Generalization of the *Whittle formula* giving a closed-form expression of the asymptotic FIM rate for *zero-mean purely non-deterministic stationary parameterized Gaussian processes* to *the more general SIRP framework.*
    - for non-centered processes
    - for purely cyclostationary processes

  This new formula has a similar structure, involving two coefficients c_0 and c_2, which depend on the texture distribution τ of the compound Gaussian (CG) process.
- Limitation of the Result: Three assumptions are required:
    - The sequence $\mathbf{R}_x(k)$ is absolutely summable.
    - The spectrum $\mathbf{S}_x(f)$ is non-singular for all f.
    - The process is Markovian with finite Markov order.
- Open questions: Can the latter assumptions be relaxed or only required? How can the results be extended to Gaussian and SIRP stationary fields?